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by

G.J. Shutts

July 1988

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Abstract

A new set of balanced equations (to be called planetary semi-geostrophic equations) for planetary-scale flow is derived from Hamilton's principle and constitutes a natural generalisation of the semi-geostrophic equations of motion. Analogues of the global conservation of energy and the Lagrangian conservation of potential vorticity follow automatically by introducing approximations directly into the Hamiltonian in such a way that time and particle label symmetries are preserved. Two approximations are required: firstly, the kinetic energy associated with the component of velocity parallel to the axis of rotation is neglected; and secondly, the Lagrangian rate of change of the wind and pressure gradient directions (when projected onto the equatorial plane) must be small compared to twice the angular rotation rate of the system. Although the first of these approximations entails some loss of accuracy for application to the terrestrial atmosphere it is not nearly as severe as that for the Phillips type II geostrophic equations in which all of the kinetic energy is omitted from the Hamiltonian.

The resulting equations take exactly the same form as the f-plane semi-geostrophic equations apart from a modification to the pseudo-density appearing in the continuity equation.

They are also amenable to the geostrophic momentum coordinate transformation - a device which has had considerable impact on the theory of atmospheric fronts. In order to assess the accuracy of the equations, two different linearised eigenvalue problems on the sphere are solved and

compared with the equivalent primitive equation problems. Eigenmodes are least accurate for high zonal wavenumber disturbances with grave meridional structure.

1. Introduction

Arguably the greatest achievement of meteorological science to date has been the development and operational use of global numerical models to forecast the weather (White et al, 1987). Although parametrization of sub-grid scale physical processes such as radiative transfer and boundary layer momentum transport render the mathematical problem extremely complicated, most of the success is directly attributable to the fidelity with which atmospheric motion is represented by the Euler equations of compressible fluid motion. The only filtering approximation it is found convenient to introduce is the hydrostatic assumption which removes all but the horizontally propagating sound wave and causes negligible error at scales currently resolvable in global models: the resulting equations are known as the primitive equation set.

Because of the diversity of solutions supported by the primitive equations, much of our theoretical understanding has been derived from solutions obtained after judicious simplification of the equations. For instance, by assuming motion on a spherical planet to be horizontally non-divergent and the air density uniform, the dynamics of planetary Rossby waves were exposed (Haurwitz, 1940). A less severe filtering approximation, based on a scale analysis, was put forward by Charney (1948) and this led to the modern quasi-geostrophic theory (see also Charney and Stern, 1962 and White, 1977) which provides the basic mathematical framework for much of our understanding of large-scale atmospheric motion (e.g. baroclinic instability theory).

Phillips (1963) identified two types of quasi-geostrophic equations; Type I being the above set due to Charney and Type II whose properties were elucidated by Burger (1958). Both types have analogues of the global energy conservation and Lagrangian potential vorticity conservation properties of the primitive equations yet neither is sufficiently accurate as a model of the entire terrestrial atmosphere or ocean. Type I is not strictly valid over a wide range of latitudes; it neglects the full variation of the Coriolis parameter except where differentiated, and linearizes the vertical advection of entropy about an assumed basic state entropy field which is dependent on height alone. In particular, the Coriolis parameter is constant in the geostrophic wind relation unlike the Type II equations. The principal difficulty with the Type II equations is the severity of the approximation to the momentum equations - all the acceleration terms are omitted and the horizontal wind is assumed to be geostrophic in the continuity of mass and thermodynamic equations.

A less restrictive class of geostrophically-balanced models which could be used with more confidence in the spherical domain was identified by Lorenz (1960). These conserve global energy though they do not have an analogue of potential vorticity conservation on fluid parcels. Numerous other forms of balanced model have been proposed and have been classified by McWilliams and Gent (1980) and reviewed by Gent and McWilliams (1983). In this paper attention will be focused on the semi-geostrophic equations proposed by Eliassen (1948) and developed through the use of the geostrophic momentum coordinate transformation by Hoskins (1975). The importance of the semi-geostrophic equations stems from their conceptual

simplicity and analytic solutions have provided much insight into frontogenesis (Hoskins and Bretherton, 1972) and flow over two-dimensional orography (Pierrehumbert, 1985). It is now known that the semi-geostrophic equations admit discontinuous solutions which 'in the real world' correspond to atmospheric fronts and inversions. An extended semi-geostrophic theory was developed by Cullen and Purser (1984) and has led to a 'geometrical' technique for solving the Lagrangian form of the semi-geostrophic equations (Cullen et al, 1987a; Chynoweth, 1987 and Shutts et al, 1988). Cullen and Purser recognised that at any instant, a fluid parcel could be characterised by the vector gradient of a modified pressure function (P). The positions of all fluid parcels within a convex region of space were then proved to be uniquely determined by the requirement that P be convex within that domain. The convexity of P corresponds to the physical necessity that the fluid parcels be arranged into a convectively and symmetrically stable state. Shutts and Cullen (1987) and Cullen et al (1987b) show this to be a minimum energy state.

The geometrical solution technique allows the implicit geostrophic adjustment inherent in the conventional semi-geostrophic theory to extend to convectively unstable situations: model elements are able to perform a type of penetrative convection. Artificial viscosity is not required and discontinuous solutions are readily described.

Salmon (1983, 1985) has shown how semi-geostrophic theory may be extended to situations where the Coriolis parameter is a function of horizontal position. Starting with a Lagrangian formulation of Hamilton's principle for the Euler equations, he introduces approximations which

preserve the symmetries of the Hamiltonian so that analogues of the conservation laws are obtained. A new set of balanced equations is obtained (Salmon, 1985) which take on a simple form in suitably chosen transformed space coordinates.

In this paper, Hamilton's principle is used to formulate a new set of balanced equations for planetary flow which have the usual f-plane semi-geostrophic equations as a special case. The procedure adopted is slightly different from Salmon's: a canonical form for the principle is sought before making any approximations to the Hamiltonian. This has the effect of making explicit the Lagrangian nature of the basic assumptions underlying the validity of semi-geostrophic theory. The geostrophic approximation does not have to be introduced directly into the Lagrangian of Hamilton's principle. It is, in fact, furnished by the variational principle itself. Also there is an explicit recognition that the planetary rotation and gravitation vectors are not collinear.

The equations derived here are different from those of Salmon and have more in common with the f-plane semi-geostrophic theory.

2. New balanced equations derived from Hamilton's principle

(a) 'Primitive equations

Following Salmon (1983), the extended form of Hamilton's principle will be used for which particle positions and conjugate momenta are independent coordinates. For a compressible, rotating fluid under the action of a gravitational field, Hamilton's principle may be written as:

$$\delta \int_{t_o}^{t_1} d\tau \left\{ \int_D \left[(u - \Omega y) \frac{\partial x}{\partial \tau} + (v + \Omega x) \frac{\partial y}{\partial \tau} + w \frac{\partial z}{\partial \tau} \right] d\Gamma - H(\tau) \right\} = 0 \qquad - (1)$$

where the Hamiltonian function H is given by:

$$H(\tau) = \int_D d\Gamma \left[{}^1/_2(u^2+v^2+w^2) + U + \Phi - p_o \left(J\left(\frac{x,y,z}{a,b,c}\right) - \alpha\right) - T_o (S - S_o)\right] \qquad - (2)$$

using a Cartesian representation (x,y,z) for the physical space coordinates of a fluid particle in a system with rotation vector (o,o,Ω) and where (u,v,w) is the velocity of the fluid, U(α,S) is the internal energy, α is the specific volume, S is the gas entropy, Φ (x,y,z) is the gravitational potential (with centrifugal term absorbed) and J is the Jacobian of the transformation between (x,y,z) and particle label space (a,b,c). Unless otherwise indicated, all dependent variables in eqs. (1) and (2) are to be regarded as functions of particle label and time τ. Therefore, we may tentatively identify (u,v,w) with

$$\left(\frac{\partial x}{\partial \tau}, \frac{\partial y}{\partial \tau}, \frac{\partial z}{\partial \tau}\right) \quad .$$

Integration over the domain (denoted by D) is with respect to particle label coordinate and $d\Gamma = da\ db\ dc = d(\text{mass})$ by definition.

Continuity of mass and conservation of entropy, represented by

$$J\left(\frac{x,y,z}{a,b,c}\right) = \alpha \qquad - (3)$$

and $S = S_o(a,b,c)$ respectively, $\qquad - (4)$

are enforced through the Lagrange multipliers $p_o(a,b,c,\tau)$ and $T_o(a,b,c,\tau)$ appearing in the Hamiltonian. Variations made in eq. (1) are such that u,v,w,x,y,z,α and S are treated as independent.

Consider, for instance, variations with respect to x; eq. (1) then gives:

$$\int_{t_o}^{t_1} d\tau \left\{\int_D \left[\delta x\left(-\frac{\partial(u-\Omega y)}{\partial \tau} + \Omega\frac{\partial y}{\partial \tau} - \frac{\partial \Phi}{\partial x}\right) - p_o\ \delta\left\{J\left(\frac{x,y,z}{a,b,c}\right)\right\}\right] d\Gamma\right\} = 0 \qquad - (5)$$

and since

$$p_o \delta J\left(\frac{x,y,z}{a,b,c}\right) = J\left(\frac{p_o \delta x, y, z}{a,b,c}\right) - \delta x J\left(\frac{p_o, y, z}{a,b,c}\right)$$

or, using eq. (3) and the rule for multiplication of Jacobians,

$$p_o \delta J\left(\frac{x,y,z}{a,b,c}\right) = J\left(\frac{p_o \delta x, y, z}{a,b,c}\right) - \delta x \alpha \frac{\partial p_o}{\partial x} \quad . \qquad - (6)$$

If δx is required to vanish on the boundary of D then the Euler-Lagrange equation derived from eq. (5) is:

$$\frac{\partial u}{\partial \tau} - 2\Omega \frac{\partial y}{\partial \tau} + \frac{\partial \Phi}{\partial x} + \alpha \frac{\partial p_o}{\partial x} = 0 \qquad - (7)$$

Similarly, independent variations of y and z give:

$$\frac{\partial v}{\partial \tau} + 2\Omega \frac{\partial x}{\partial \tau} + \frac{\partial \Phi}{\partial y} + \alpha \frac{\partial p_o}{\partial y} = 0 \qquad - (8)$$

and $$\frac{\partial w}{\partial \tau} + \frac{\partial \Phi}{\partial z} + \alpha \frac{\partial p_o}{\partial z} = 0 \qquad - (9)$$

Furthermore, independent variations with respect to the remaining functions u, v, w, α and S give:

δu: $$\frac{\partial x}{\partial \tau} = u \qquad - (10)$$

δv: $$\frac{\partial y}{\partial \tau} = v \qquad - (11)$$

δw: $$\frac{\partial z}{\partial \tau} = w \qquad - (12)$$

$\delta \alpha$: $$p_o = \frac{\partial U}{\partial \alpha} \qquad - (13)$$

and

δS: $$T_o = \frac{\partial U}{\partial S} \quad . \qquad - (14)$$

Eqs.(10) - (12) provide no new information but merely demonstrate self-consistency of the definitions; eqs (12) and (13) show that p_o and T_o are simply the pressure and temperature - consistent with their thermodynamic definition. Therefore, eqs (7) - (9) may be written vectorially, with the aid of (10) -(14), as:

$$\frac{D\underline{V}}{Dt} + 2\,\underline{\Omega} \wedge \underline{V} + \underline{\nabla}\Phi + \alpha\,\underline{\nabla}p = 0 \qquad - (15)$$

where $\frac{\partial}{\partial\tau} \equiv \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{V}.\underline{\nabla}$

$$t = \tau$$

$$\underline{V} = (u,v,w)$$

$$\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\underline{\Omega} = (0,0,\Omega)$$

and all dependent variables are regarded as functions of (x,y,z,t). Eq. (15) is the Euler equation of motion for an inviscid, rotating fluid.

The meteorological primitive equations can be obtained for the f-plane case (where $\underline{\nabla}\Phi \parallel \underline{\Omega}$) by defining $\Phi = gz$ and omitting all terms in w from eq. (1). This removes the geopotential terms from eqs. (7) and (8) and reduces eq. (9) to the hydrostatic equation:

$$g + \alpha \frac{\partial p}{\partial z} = 0 \qquad . \qquad - (16)$$

Extension to planetary flows where the gravitation and rotation vectors are not collinear is achieved by expressing eq. (1) in spherical polar coordinates; omitting terms in Dr/Dt (the local vertical velocity); expanding the radial coordinate, r about a mean planetary radius a and neglecting terms of the order of (scale height of the atmosphere)/a.

The hydrostatic approximation is valid, therefore, if the kinetic energy is well approximated by that associated with the horizontal components of the flow. If h is a typical depth scale and l a typical horizontal length scale of an atmospheric motion system, then conventional scale analysis requires that $h^2/l^2 \ll 1$ for the validity of the hydrostatic assumption (Holton, 1979). Whilst this inequality correctly identifies hydrostatic motion in the terrestrial atmosphere, it is not a necessary condition as was pointed out by Phillips (1963). The Hamiltonian formulation of the primitive equations emphasises the dependence of the hydrostatic assumption on the smallness of the kinetic energy in the vertical motion.

As shown by Salmon (1983, 1985 and 1987), approximations that preserve the time and particle labelling symmetries of the Lagrangian in Hamilton's principle automatically imply that the Hamiltonian is an integral invariant and that an analogue of potential vorticity conservation exists. These properties will be considered later for the new equations to be derived.

(b) Phillips Type II Quasi-geostrophic equations

The 'potential' energy term $U + \Phi$ in eq. (1) can be partitioned into a basic state component (for which the gas entropy is a function of pressure alone) and an available potential energy component which can be converted into kinetic energy (Lorenz, 1955). For some scales of motion the available potential energy may dominate the kinetic energy; this happens when the Burger number B_u given by:

$$B_u = \left(\frac{\bar{N}h}{2\Omega l}\right)^2$$

satisfies $B_u << 1$ where $\bar{N}$ is a mean buoyancy frequency (Burger, 1958). In the atmosphere this assumption is only valid for the largest scales of planetary Rossby wave motion; on the other hand extensive regions of the ocean have small Burger number.

Motivated by the smallness of the kinetic energy in these situations, all terms in eq. (1) involving u,v or w may be removed leaving:

$$\delta \int_{t_o}^{t_1} d\tau \left\{ \int_D \Omega \left(x \frac{\partial y}{\partial \tau} - y \frac{\partial x}{\partial \tau} \right) d\Gamma - H(\tau) \right\} = 0 \qquad - (17)$$

where $H(\tau) = \int_D d\Gamma \left[U + \Phi - p_o \left\{ J\left(\frac{x,y,z}{a,b,c}\right) - \alpha \right\} - T_o(S - S_o) \right]$. $\qquad - (18)$

The Lagrangian in eq. (17) is closely related to that in eq. 4.1 of Salmon (1983) which was for a shallow water fluid system with a spatially varying Coriolis parameter and rigid boundary at z = o. Independent variations with respect to x, y and z give:

$$\delta x: \quad 2\Omega \frac{\partial y}{\partial \tau} - \frac{\partial \Phi}{\partial x} - \alpha \frac{\partial p}{\partial x} = 0 \qquad - (19)$$

$$\delta y: \quad -2\Omega \frac{\partial x}{\partial \tau} - \frac{\partial \Phi}{\partial y} - \alpha \frac{\partial p}{\partial y} = 0 \qquad - (20)$$

$$\delta z: \quad - \frac{\partial \Phi}{\partial z} - \alpha \frac{\partial p}{\partial z} = 0 \qquad - (21)$$

where, as before, p_o is identified with pressure p. Equations (14) - (21) may be combined to give:

$$2\,\underline{\Omega} \wedge \underline{V} + \underline{\nabla}\Phi + \alpha\,\underline{\nabla}p = 0 \qquad - (22)$$

which is a vector statement of geostrophic and hydrostatic balance. The Eulerian forms of the continuity and thermodynamic equations are:

$$\left(\frac{\partial}{\partial t} + \underline{V}.\underline{\nabla} \right) \alpha = \alpha\, \underline{\nabla}.\underline{V} \qquad - (23)$$

and

$$\left(\frac{\partial}{\partial t} + \underline{V}.\underline{\nabla} \right) S = 0 \qquad - (24)$$

which, together with eq. (22), the definition of S and the perfect gas equation, form a closed set of equations. Apart from the inclusion of a small Coriolis term associated with the horizontal component of $\underline{\Omega}$, the

spherical polar expression of this set is widely known as the Phillips Type II quasi-geostrophic equations or, the Burger equations. It is convenient to express them in spherical polar coordinates because Φ is radially symmetric and atmospheres are highly stratified in the radial direction. A consequence of this stratification is that $\underline{V}$ is dominated by its horizontal components and a unique geostrophic wind $\underline{V}_g$ can be defined such that:

$$2(\underline{k}.\underline{\Omega})\underline{V}_g = \alpha\, \underline{k} \wedge \underline{\nabla} p \qquad - (25)$$

where $\underline{k}$ is the local unit vector in the direction of $\underline{\nabla}\Phi$ (Phillips, 1963) and $\underline{V}_g$ may be used in eqs. (23) and (24) in place of $\underline{V}$. (Note that eq. (22) does not, by itself, define $\underline{V}$ uniquely).

An alternative and illuminating approach is to express eqs. (22-24) in (X,Y,Z) coordinates where

$$x = X$$
$$y = Y$$
$$Z = Z_a \left[1 - (p/p_*)^{(\gamma-1)/\gamma}\right] \qquad - (26)$$

where γ is the ratio of specific heats ($^{C_p}/_{C_v}$), p_* is a constant reference pressure taken here to be mean sea-level pressure, Z_a is given by

$$Z_a = \frac{\gamma p_* \alpha_o}{(\gamma-1)g},$$

g is the acceleration due to gravity and α_* is the mean specific volume at p_*. The partial derivatives can then be shown to transform according to:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \Pi(p) \frac{\partial p}{\partial x} \frac{\partial}{\partial Z} \qquad \text{(similarly for y)} \qquad - (27)$$

and $$\frac{\partial}{\partial z} = \Pi(p) \frac{\partial p}{\partial z} \frac{\partial}{\partial Z} \qquad - (28)$$

where $$\Pi(p) = Z_a \frac{(1 - \gamma)}{\gamma p_*} \left(\frac{p}{p_*}\right)^{-1/\gamma} .$$

Using eqs (21) (27) and (28) we obtain:

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial X} + \Pi (p) \frac{\partial p}{\partial x} \frac{\partial \Phi}{\partial Z} \qquad - (29)$$

$$\frac{\partial \Phi}{\partial z} = \Pi (p) \frac{\partial p}{\partial z} \frac{\partial \Phi}{\partial Z} = - \alpha \frac{\partial p}{\partial z} \qquad - (30);$$

(30) implies that:

$$\Pi(p) \frac{\partial \Phi}{\partial Z} = - \alpha \qquad - (31)$$

which, on substitution into eq. (29) gives:

$$\frac{\partial \Phi}{\partial X} = \frac{\partial \Phi}{\partial x} + \alpha \frac{\partial p}{\partial x} \qquad - (32)$$

and similarly for the y component.

Equations (19) and (20) may now be simplified to:

$$2 \Omega v = \frac{\partial \Phi}{\partial X} \qquad - (33)$$

and $$2 \Omega u = - \frac{\partial \Phi}{\partial Y} \qquad - (34)$$

and using the perfect gas equation and the definition of potential temperature θ, eq. (31) may be rewritten as:

$$\frac{\partial \Phi}{\partial Z} = \frac{g\theta}{\theta_o} \qquad - (35)$$

where $\theta_o = p_* \alpha_* / R$ and R is the gas constant. Hoskins and Bretherton (1972) introduced the pressure-dependent local vertical coordinate Z because it combines some of the advantages of pressure coordinates (anelastic continuity equation, simplified pressure gradient force terms) with its tendency to approximate physical height in the troposphere and give a hydrostatic relation involving potential temperature. Cartesian f-plane problems solved using this transformation invariably have horizontal z surfaces which are quasi-parallel to Z surfaces so that horizontal differentiation holding Z constant can be visualized as virtually the same as holding z constant. In its application to planetary flows here, z surface are <u>not</u> horizontal and $\partial/\partial x$ is vastly different from $\partial/\partial X$ (Fig. 1(a)) due to the near-sphericity of Z surfaces. Similarly, one should be aware of the difference between $\partial/\partial Z$ holding X and Y constant and the usual vertical derivative in Z coordinates. We may refer to $\partial/\partial Z$ as an axial derivative to distinguish it from differentiation with respect to Z in the direction of $\underline{\nabla}\Phi$ (the local vertical, see Fig. 1 (b)). It is easy to show from Fig. 1(b) that:

$$\frac{\partial}{\partial Z} = \left(\frac{\partial}{\partial Z}\right)_{LV} + \frac{\cot\phi}{r} \frac{\partial}{\partial \phi} \qquad - (36)$$

where ϕ is latitude, r is the distance from the earth's centre and LV denotes differentiation in the direction of the local vertical. This implies (using eq. 35) a hydrostatic relation of the form.

$$- \frac{g\theta}{\theta_o} + \left(\frac{\partial \Phi}{\partial Z}\right)_{LV} = - \frac{\cot\phi}{r} \frac{\partial \Phi}{\partial \phi} = - 2\,\Omega \cos\phi\, u_g \qquad - (37)$$

where u_g is the conventional zonal geostrophic wind speed. It is noteworthy that the axial hydrostatic balance (eq. 35) implies a radial balance which includes the small Coriolis term due to zonal motion.

Consider now the continuity equation (23) in a Cartesian space whose coordinates are (X,Y,Z). This is simply

$$\frac{\partial \rho'}{\partial T} + \frac{\partial(\rho' u)}{\partial X} + \frac{\partial(\rho' v)}{\partial Y} + \frac{\partial(\rho' w)}{\partial Z} = 0 \qquad - (38)$$

where $w = \frac{DZ}{DT}$ $(\frac{D}{DT} \equiv \frac{\partial}{\partial \tau})$ and ρ' is the density in (X,Y,Z) space given by

$$\rho' = \alpha^{-1} J\left(\frac{x,y,z}{X,Y,Z}\right) .$$

But $J\left(\frac{x,y,z}{X,Y,Z}\right) = \frac{\partial z}{\partial Z} = \left(\Pi(p) \frac{\partial p}{\partial z}\right)^{-1}$

and therefore:

$$\rho' = \left(\Pi(p)\, \alpha \frac{\partial p}{\partial z}\right)^{-1} = \rho(Z)\left(\frac{1}{g} \frac{\partial \Phi}{\partial z}\right)^{-1} \qquad - (39)$$

where $\rho(Z)$ is the pseudo-density defined in Hoskins and Bretherton (1972) as:

$$\rho(Z) = \alpha_o^{-1} (p/p_*)^{1/\gamma} . \qquad - (40)$$

Now $\partial\Phi/\partial z = g \sin\phi \, [1 + O(h/a)]$ where ϕ is latitude and a is the mean radius of the earth and so:

$$\rho' = \frac{\rho(Z)}{\sin\phi} . \qquad - (41)$$

Also $\mathrm{Sin}\phi = \{1 - (X^2+Y^2)/a^2 + 0\ (h/a)\}^{1/2}$ and since $h/a \sim 10^{-3}$, $\mathrm{Sin}\phi$ may be regarded as a function of X and Y, so that $\rho' = \rho'(X,Y,Z)$. The continuity equation in transformed coordinates (eq. 36 then assumes the anelastic form:

$$\frac{\partial(\rho' u)}{\partial X} + \frac{\partial(\rho' v)}{\partial Y} + \frac{\partial(\rho' w)}{\partial Z} = 0 \qquad - (42)$$

which, on substitution for u and v from eqs (33) and (34), becomes:

$$\frac{1}{2\Omega} J\left(\frac{\rho', \Phi}{X, Y}\right) = \frac{\partial(\rho' w)}{\partial Z} \quad . \qquad - (43)$$

To compare this equation with that derived by Phillips (1963), a transformation of X and Y to latitude ϕ and longitude λ is required. The mapping

$$X = a \cos\lambda \cos\phi + 0\ (h/a)$$

$$Y = a \sin\lambda \cos\phi + 0\ (h/a)$$

has a Jacobian of transformation given by:

$$J\left(\frac{X, Y}{\lambda, \phi}\right) = a^2 \cos\phi \sin\phi + 0\ (h/a) \quad . \qquad - (44)$$

Multiplying eq. (43) by this Jacobian and simplifying gives:

$$J\left(\frac{\rho', \Phi}{\lambda, \phi}\right) = - \frac{\partial\Phi}{\partial\lambda} \frac{\partial}{\partial\phi}\left[\frac{\rho(Z)}{\sin\phi}\right] = 2\,\Omega\, a^2 \cos\phi \sin\phi \frac{\partial(\rho' w)}{\partial Z} \qquad - (45)$$

which, on using eqs. (36) and (41), gives

$$\frac{1}{\mathrm{Sin}^2\phi}\,\frac{\partial\Phi}{\partial\lambda} = 2\Omega a^2\left[\left\{\frac{1}{\rho}\frac{\partial}{\partial Z}(\rho w)\right\}_{LV} + \frac{\mathrm{Cot}\phi}{a}\,\frac{\partial w}{\partial\phi} - \frac{w}{a}\,\mathrm{Cot}^2\phi\right] . \qquad - (46)$$

This equation is the compressible equivalent of eq. (6.3) on pg. 163 of Phillips' review except for the second and third terms on the right-hand side of the equation. As alluded to earlier, these result from the inclusion of a small Coriolis term. To show this eq. (46) may be derived by substituting the following expressions for u_g and v_g

$$2\Omega\mathrm{Sin}\phi.u_g = -\frac{1}{a}\frac{\partial\Phi}{\partial\phi} \qquad - (47)$$

$$2\,\Omega\mathrm{Sin}\phi.v_g \quad \frac{1}{a\mathrm{Cos}\phi}\,\frac{\partial\Phi}{\partial\lambda} + 2\Omega\mathrm{Cos}\phi w \qquad - (48)$$

into the continuity equation:

$$\frac{2w}{a} + \frac{1}{\rho}\left\{\frac{\partial(\rho w)}{\partial Z}\right\}_{LV} + \frac{1}{a\mathrm{Cos}\phi}\left\{\frac{\partial(v_g\mathrm{Cos}\phi)}{\partial\phi} + \frac{\partial u_g}{\partial\lambda}\right\} = 0 \qquad - (49)$$

noting the inclusion of the tiny metric term $2w/a$ for air parcels moving radially.

Since $S = C_p \ln\theta$, the thermodynamic equation (24) may be written as:

$$\frac{\partial\theta}{\partial T} + \frac{1}{2\Omega}\,J\left(\frac{\Phi,\theta}{X,Y}\right) + w\frac{\partial\theta}{\partial Z} = 0$$

or, using eqs (44) and (35):

$$\frac{\partial^2\Phi}{\partial T\partial Z} + \frac{1}{2\Omega a^2\,\mathrm{Sin}\,\phi\,\mathrm{Cos}\,\phi}\,J\left(\frac{\Phi,\frac{\partial\Phi}{\partial Z}}{\lambda,\phi}\right) + w\frac{\partial^2\Phi}{\partial Z^2} = 0 \qquad - (50)$$

Eqs (45) and (50) form a closed set involving Φ and w.

Although, as we have seen, these are a very severely approximated balanced set, they are widely used in oceanography (eg. Pedlosky (1979); Anderson and Killworth, 1979) and have, on occasion, been used to study ultra-long waves in the atmosphere, (e.g. Bates, 1977; Lynch (1979)).

Global energy and Lagrangian potential vorticity conservation follow from time and particle label symmetries. The details are not reproduced here though the method is essentially the same as in 2(d).

c. New 'planetary semi-geostrophic' equations

Without any regard for accuracy at this stage, the first approximation to be made is to omit all terms in w from eq. (1). As was seen in a., this gives the hydrostatic approximation for an f-plane system whose axis of rotation is in the z-direction. Hamilton's principle then requires that:

$$\delta \int_{t_o}^{t_1} d\tau \left\{ \int_D \left[(u-\Omega y)\frac{\partial x}{\partial \tau} + (v+\Omega x)\frac{\partial y}{\partial \tau} \right] d\Gamma - H(\tau) \right\} = 0 \qquad - (51)$$

where now:

$$H(\tau) = \int_D d\Gamma \left[\frac{1}{2}(u^2 + v^2) + U + \Phi - p_o\left(J\left(\frac{x,y,z}{a,b,c}\right) - \alpha\right) - T_o\,(S - S_o) \right] \qquad - (52)$$

We now seek a transformation to canonical form by letting:

$$(u - \Omega y)\frac{\partial x}{\partial \tau} + (v + \Omega x)\frac{\partial y}{\partial \tau} = \frac{1}{2\omega}\left(M\frac{\partial N}{\partial \tau} - N\frac{\partial M}{\partial \tau}\right) + \frac{\partial B}{\partial \tau} + C \qquad - (53)$$

where M and N are to be the new canonical coordinates, ω is an unknown constant, B is an arbitrary function of the old and new coordinates and C represents the residual terms which, when neglected, define the approximation.

Inspired by the canonical form of eq. (17) which leads to the Burger equations, we choose:

$$M = \omega x + F(a,b,c,\tau)$$
$$N = \omega y + G(a,b,c,\tau) \qquad - (54)$$

where F and G are unknown functions to be determined. Substitution of eqs (54) into eq. (53) suggests the choice:

$$\omega = 2\Omega \ , \ G = -u \text{ and } F = v$$

which, on rearrangement, gives:

$$(u - \Omega y)\frac{\partial x}{\partial \tau} + (v + \Omega x)\frac{\partial y}{\partial \tau} = \frac{1}{4\Omega}\left(M\frac{\partial N}{\partial \tau} - N\frac{\partial M}{\partial \tau}\right) + \frac{\partial}{\partial \tau}\frac{(xu+yv)}{2} - \frac{1}{4\Omega}\left(u\frac{\partial v}{\partial \tau} - v\frac{\partial u}{\partial \tau}\right)$$

with $M = 2\Omega x + v$ and $N = 2\Omega y - u$. $\qquad - (55)$

But the last term in eq. (55) may be written as:

$$\frac{u^2 + v^2}{4\Omega}\frac{\partial \chi}{\partial \tau}$$

where χ is the angle (measured anticlockwise) that the velocity vector, projected onto the xy plane, makes with an arbitrary fixed direction. Since in Hamilton's principle, variations are taken to vanish at the endpoint times t_o and t_1 the term $\partial/\partial\tau(xu+yv)/2$ integrates out and eqs (51) and (52) become:

$$\delta\int_{t_o}^{t_1} d\tau \left\{\frac{1}{4\Omega}\int\left[M\frac{\partial N}{\partial\tau} - N\frac{\partial M}{\partial t}\right]d\Gamma - H'(\tau)\right\} = 0 \qquad - (56)$$

$$H'(\tau) = \int_D\left\{\frac{1}{2}\{(2\Omega y - N)^2 + (M - 2\Omega x)^2\}\left(1 + \frac{1}{2\Omega}\frac{\partial\chi}{\partial\tau}\right) + U + \Phi - p_o\left(J\left(\frac{x,y,z}{a,b,c}\right) - \alpha\right)\right.$$

$$\left. - T_o(S - S_o)\right]d\Gamma \; . \qquad - (57)$$

Our second approximation requires that the term $(1/2\Omega)\partial\chi/\partial\tau$ be neglected under the assumption that:

$$\frac{1}{2\Omega}\left|\frac{\partial\chi}{\partial\tau}\right| << 1 \quad . \qquad (A)$$

Physically, this implies that the rate of turning of the wind vector following fluid particles (in the xy plane) is small compared to twice the angular rotation rate of the system. Alternatively, it requires that the centrifugal force of the relative motion be small compared to the Coriolis force.

Independent variations of M,N,x,y and z are easily seen to give:

$$\delta M: \qquad \frac{\partial N}{\partial\tau} = 2\Omega\,(M - 2\Omega x) \qquad - (58)$$

$$\delta N: \qquad \frac{\partial M}{\partial\tau} = 2\Omega\,(2\Omega y - N) \qquad - (59)$$

$$\delta x: \qquad 2\Omega v = 2\Omega(M-2\Omega x) = \frac{\partial \Phi}{\partial x} + \alpha \frac{\partial p}{\partial x} \qquad - (60)$$

$$\delta y: \qquad -2\Omega u = 2\Omega(2\Omega y - N) = \frac{\partial \Phi}{\partial y} + \alpha \frac{\partial p}{\partial y} \qquad - (61)$$

$$\delta z: \qquad \frac{\partial \Phi}{\partial z} + \alpha \frac{\partial p}{\partial z} = 0 \qquad - (62) \ .$$

For condition (A) to be consistent 'after the fact' we require, using (60) and (61), that:

$$\frac{1}{2\Omega} \left| \frac{\partial \hat{v}}{\partial \tau} \right| << 1 \qquad - (B)$$

where $\hat{v}$ is the angle (measured anti-clockwise) that the vector $\underline{\nabla} \Phi + \alpha \underline{\nabla} p$ makes with any fixed direction when projected into the equatorial plane. Conditions (A) and (B) will be treated as independent since (A) is required to obtain the Euler equations (58)-(62) in the first instance and (B) is implied on substitution for u and v from eqs (60) and (61).

Introducing the coordinate transformation described in (b) to simplify eqs (60) - (62) gives:

$$2\Omega(M-2\Omega X) = \frac{\partial \Phi}{\partial X} \qquad - (63)$$

$$-2\Omega(2\Omega Y - N) = \frac{\partial \Phi}{\partial Y} \qquad - (64)$$

$$\frac{g\theta}{\theta_o} = \frac{\partial \Phi}{\partial Z} \qquad - (65)$$

and the evolution equations (58) and (59) may be written as:

$$\frac{DN}{Dt} = \frac{\partial \Phi}{\partial X} \qquad - (66)$$

$$\frac{DM}{Dt} = -\frac{\partial \Phi}{\partial Y} \qquad - (67)$$

where $^{D}/_{Dt}$ is the material derivative.

Eqs (63)-(67), together with the continuity of mass and thermodynamic equations, are identical to the f-plane semi-geostrophic equations of Hoskins (1975). In that case, the velocity component whose contribution to the kinetic energy in the Hamiltonian is neglected is directed along the local vertical. This enforces the conventional hydrostatic assumption: the accuracy of the semi-geostrophic equations then depends primarily on the smallness of particle accelerations with respect to the Coriolis force.

Condition (A) is one of two conditions Hoskins identifies as being necessary for the validity of the semi-geostrophic equations. His second condition requires that:

$$|DV| \ll fV$$

where $V = |\underline{V}|$ and $D \equiv D/Dt$ and can be shown to be implied by conditions (A) and (B) as follows. The unapproximated momentum equation may be expressed in natural coordinates ξ and η, along and across the flow respectively, giving:

$$DV + \Phi_{\xi} = 0 \qquad - (68)$$

$$- VD\chi - fV + \Phi_\eta = 0 \qquad - (69)$$

(see Hoskins, 1975). If β is the angle made between the horizontal projection of the geopotential gradient and the η direction so that:

$$\text{Sin}\,\beta = \left|\frac{\Phi_\xi}{\underline{\nabla}\phi}\right| \quad \text{and} \quad \text{Cos}\beta = \left|\frac{\Phi_\eta}{\underline{\nabla}\phi}\right|$$

then: $D\beta = \text{Cos}^2\beta D(\Phi_\xi/\Phi_\eta)$. $\qquad - (70)$

Consistent with condition (A), $|D\chi/f| \ll 1$ and eq. (69) imply that:

$$fV = \phi_\eta. \qquad - (71)$$

Using eqs (68) and (71), eq. (70) may be written as:

$$D^*\beta = - \text{Cos}^2\beta\ D^*(V^{-1}D^*V)$$

where $D^* = D/f$, but:

$D^*\hat{v} = D^*(\chi + \beta)$ so that:

$$D^*\hat{v} = D^*\chi - \text{Cos}^2\beta D^*(V^{-1}D^*V) \quad . \qquad - (72)$$

In general, conditions (A) and (B) ($D^*\chi \ll 1$ and $D^*\hat{\nu} \ll 1$) then demand that:

$$|D^*(V^{-1}D^*V)| \ll 1$$

or $$\left|\frac{D^*V}{V}\right| \ll 1$$ which is Hoskins' second condition.

Neither of conditions (A) and (B) alone suffices to ensure the validity of semi-geostrophic (SG) theory. For instance, the SG equations admit coastal Kelvin wave modes in which the flow is unidirectional ($D^*\chi=0$) and yet the corresponding isobars are not parallel to the coast. SG Kelvin waves are only accurate for long wavelengths corresponding to $D^*\hat{\nu} \ll 1$. On the other hand SG flow forced by a purely two-dimensional diabatic heat source generates a unidirectional pressure gradient so that $D^*\hat{\nu} = 0$ and yet $D^*\chi \neq 0$. The SG equations are only valid provided that the heat source is switched on sufficiently slowly that $D^*\chi \ll 1$.

Planetary scale flows are also governed by eqs (63)-(67) but the absence of the velocity component parallel to the axis of rotation from the kinetic energy renders the system relatively less accurate than the f-plane semi-geostrophic equations. Under these circumstances the system (63)-(67 will be called the planetary semi-geostrophic equations. Only planetary motions which are zonally elongated or have small Burger number will be accurately treated. An attempt to quantify the distortion inherent in these planetary semi-geostrophic equations with respect to the primitive equations is described in Section 3.

Although the PSG equations are extremely simple in the Cartesian form (63)-(67), they are - for some practical purposes - better expressed in spherical polar coordinates. As in Section (b), let:

$$X = a\,\mathrm{Cos}\lambda\,\mathrm{Cos}\phi$$

$$Y = a\,\mathrm{Sin}\lambda\,\mathrm{Cos}\phi$$

so that:

$$\frac{\partial\Phi}{\partial X} = -\frac{1}{a}\,\frac{\mathrm{Sin}\lambda}{\mathrm{Cos}\phi}\,\frac{\partial\Phi}{\partial\lambda} - \frac{1}{a}\,\frac{\mathrm{Cos}\lambda}{\mathrm{Sin}\phi}\,\frac{\partial\Phi}{\partial\phi} \qquad - (73)$$

$$\frac{\partial\Phi}{\partial Y} = \frac{\mathrm{Cos}\lambda}{a\mathrm{Cos}\phi}\,\frac{\partial\Phi}{\partial\lambda} - \frac{1}{a}\,\frac{\mathrm{Sin}\lambda}{\mathrm{Sin}\phi}\,\frac{\partial\Phi}{\partial\phi} \qquad - (74)$$

(NB. $\partial/\partial\lambda$ and $\partial/\partial\phi$ are at constant Z) and

$$\dot{X} = -a\mathrm{Sin}\lambda\,\mathrm{Cos}\phi\,\dot{\lambda} - a\,\mathrm{Sin}\phi\,\mathrm{Cos}\lambda\,\dot{\phi} \qquad - (75)$$

$$\dot{Y} = a\mathrm{Cos}\lambda\,\mathrm{Cos}\phi\,\dot{\lambda} - a\,\mathrm{Sin}\lambda\,\mathrm{Sin}\phi\,\dot{\phi} \qquad - (76)$$

where λ and ϕ may be identified with longitude and latitude respectively to a high degrees of accuracy.

Now equations (66) and (67) may (using eqs (63) and (64)) be written as:

$$\frac{1}{2\Omega}\,\frac{D}{Dt}\,\frac{\partial\Phi}{\partial X} + 2\Omega\dot{X} = -\frac{\partial\Phi}{\partial Y} \qquad - (77)$$

and $\frac{1}{2\Omega}\frac{D}{Dt}\frac{\partial\Phi}{\partial X} + 2\Omega\dot{Y} = \frac{\partial\Phi}{\partial Y}$ - (78)

Multiplying (77) by Sinλ and substracting from this (78) x Cosλ gives:

$$\frac{D}{Dt}\left[\frac{-\Phi_\lambda}{2\Omega a\ \mathrm{Cos}\phi}\right] + \frac{\dot{\lambda}}{2\Omega a\ \mathrm{Sin}\phi}\cdot\Phi_\phi - 2\Omega a\ \mathrm{Cos}\phi\dot{\lambda} = \frac{\Phi_\phi}{a\mathrm{Sin}\phi} \qquad - (79)$$

on using eqs. (73)-(76). If u_g and v_g are the usual components eastward and northward components of the geostrophic wind given by:

$$v_g = \frac{\Phi_\lambda}{2\Omega a\ \mathrm{Sin}\phi\ \mathrm{Cos}\phi} \qquad \text{and}$$

$$u_g = -\frac{\Phi_\phi}{2\Omega a\ \mathrm{Sin}\phi} \quad \text{then eq. (79) may be written as:}$$

$$\mathrm{Sin}\phi\ \frac{D(v_g\ \mathrm{Sin}\phi)}{Dt} + \frac{u_g u'\mathrm{Tan}\phi}{a} + 2\Omega u'\ \mathrm{Sin}\phi = 2\Omega u_g\ \mathrm{Sin}\phi \qquad - (80)$$

where the prime on u indicates the actual eastward velocity component. Similarly, it can be readily shown that the zonal component of the momentum equation becomes:

$$\frac{Du_g}{Dt} - \frac{u'v_g\ \mathrm{Tan}\phi}{a} - 2\Omega\ \mathrm{Sin}\phi\ v' = -2\Omega\ \mathrm{sin}\phi\ v_g \qquad - (81)$$

Neglecting the Coriolis term in eq. (37), the vertical momentum equation expresses the conventional hydrostatic balance:

$$\frac{g\theta}{\theta_o} = \left(\frac{\partial\Phi}{\partial Z}\right)_{LV} . \qquad - (82)$$

To arrive at eqs (80) and (81) directly from the primitive equations of motion, not only does one have to introduce the geostrophic momentum assumption but also include $\mathrm{Sin}\phi$ factors in the material derivative of the meridional momentum. This is consistent with the neglect of a contribution $v_g \mathrm{Cos}\phi$ from the kinetic energy term in the Hamiltonian (i.e. the balanced kinetic energy for the equation set is $^1/_2 [u_g^2 + (v_g \mathrm{Sin}\phi)^2]$).

(d) Time and particle label symmetry

The absence of terms in the integrand of eq. (56) with an explicit time dependence means that the action defined by:

$$\int_{t_o}^{t_1} L \left(M, N, \frac{\partial M}{\partial \tau} \frac{\partial N}{\partial \tau} x, y, z, \alpha, S\right) d\tau$$

where $$L = \frac{1}{4\Omega} \int_D \left[M \frac{\partial N}{\partial \tau} - N \frac{\partial M}{\partial \tau}\right] d\Gamma - H'(\tau)$$

is invariant with respect to a relabelling of the time coordinate. Consider a new time coordinate τ' related to τ by:

$$\tau' = \tau + \delta\tau(\tau)$$

such that:

$$\delta\tau(t_1) = \delta\tau(t_o) = 0$$

and in the limit $\delta\tau \to 0$.

Following Salmon (1983), the action difference between two identical realizations with time coordinates τ and τ' is:

$$\int_{t_o}^{t_1} d\tau' L\left(M,N,\frac{\partial M}{\partial \tau'}, \frac{\partial N}{\partial \tau'}, x,y,z,\alpha,s\right) - \int_{t_o}^{t_1} d\tau L\left(M,N,\frac{\partial M}{\partial \tau}, \frac{\partial N}{\partial \tau}, x,y,z,\alpha,s\right)$$

$$= \int_{t_o}^{t_1} d\tau \frac{d\tau'}{d\tau} L\left(M,N,\frac{\partial M}{\partial \tau}\frac{d\tau}{\partial \tau'}, \frac{\partial N}{\partial \tau}\frac{d\tau}{\partial \tau'}, x,y,z,\alpha,s\right) - \int_{t_o}^{t_1} L d\tau$$

$$= \int_{t_o}^{t_1} d\tau \left(\frac{d\delta\tau}{d\tau}\right)\left[L - \int_D \left\{\delta\frac{\delta L}{(\partial M/\partial \tau)}\frac{\partial M}{\partial \tau} + \frac{\delta L}{\delta(\partial N/\partial \tau)}\frac{\partial N}{\partial \tau}\right\} d\Gamma\right] + O(\delta\tau^2)$$

$$= -\int_{t_o}^{t_1} d\tau \left(\frac{d\delta\tau}{d\tau}\right) H' + O(\delta\tau^2)$$

$$= \int_{t_o}^{t_1} d\tau \delta\tau \frac{dH'}{d\tau} + O(\delta\tau^2) = 0$$

and so, since $\delta\tau$ is arbitrary:

$$\frac{dH'}{d\tau} = 0$$

or $$\int_D \left[\frac{1}{2}(2\Omega y - N)^2 + \frac{1}{2}(M - \Omega x)^2 + U + \Phi\right] d\Gamma = \text{a constant} \qquad - (82)$$

Consider now a relabelling of the particles given by:

$$a' = a + \delta a(a,b,c,\tau)$$

$$b' = b + \delta b(a,b,c,\tau)$$

$$c' = c + \delta c(a,b,c,\tau)$$

such that the specific volume and entropy are unaltered ie:

$$\delta\alpha = \delta J\left(\frac{a,b,c}{x,y,z}\right) = 0 \qquad - (83)$$

$$\delta S = \frac{\partial S}{\partial a}\,\delta a + \frac{\partial S}{\partial b}\,\delta b + \frac{\partial S}{\partial c}\,\delta c = 0 \qquad - (84)$$

where δ denotes the change due to relabelling. Equation (84) can be readily satisfied if c is chosen so that $S = S(c)$ thereby implying that $\delta c = 0$: the relabelling therefore involves assigning new a and b labels within S surfaces. It then follows from eq. (83) that:

$$\frac{\partial \delta a}{\partial a} + \frac{\partial \delta b}{\partial b} = 0 \qquad - (85)$$

which suggests the definition of a perturbation streamfunction $\delta\psi$ given by:

$$\delta a = -\frac{\partial \delta\psi}{\partial b} \text{ and } \delta b = \frac{\partial \delta\psi}{\partial a} \qquad - (86)$$

Time differentiation is affected by the relabelling since

$$\left.\frac{\partial M}{\partial \tau}\right|_{a,b,c} = \left.\frac{\partial M}{\partial \tau}\right|_{a',b',c} + \frac{\partial M}{\partial a'}\frac{\partial a'}{\partial \tau} + \frac{\partial M}{\partial b'}\frac{\partial b'}{\partial \tau}$$

so that:

$$\delta\left(\frac{\partial M}{\partial \tau}\right) = -\frac{\partial M}{\partial a}\frac{\partial \delta a}{\partial \tau} - \frac{\partial M}{\partial b}\frac{\partial \delta b}{\partial \tau} \qquad - (87)$$

and similarly for $\frac{\partial N}{\partial \tau}$.

The Lagrangian $L(M,N,\frac{\partial M}{\partial \tau}\ \frac{\partial N}{\partial \tau}\ x,y,z,\alpha,S)$ has no explicit dependence on a

and b implying that

$$\delta\int_{t_o}^{t_1} L d\tau = \int_{t_o}^{t_1} d\tau \left\{\int_D \left[M\delta\left(\frac{\partial N}{\partial \tau}\right) - N\delta\left(\frac{\partial M}{\partial \tau}\right)\right] d\Gamma\right\}$$

which, on using eqs (86) and (87), becomes:

$$\int_{t_o}^{t_1} d\tau \left\{\int_D \left[-\frac{\partial \delta\psi}{\partial b}\frac{\partial}{\partial \tau}\left[M\frac{\partial N}{\partial a} - N\frac{\partial M}{\partial a}\right] + \frac{\partial \delta\psi}{\partial a}\frac{\partial}{\partial \tau}\left[M\frac{\partial N}{\partial b} - N\frac{\partial M}{\partial b}\right]\right] d\Gamma\right\} \qquad - (88)$$

Integrating by parts and setting $\delta\psi = 0$ on the boundary of D leads to:

$$\int_{t_o}^{t_1} d\tau \left\{\int_D \delta\psi \frac{\partial}{\partial \tau} J\left(\frac{M,N}{a,b}\right) d\Gamma\right\}$$

which is zero if the evolution of the system is to be unperturbed by relabelling particles. Since $\delta\psi$ is arbitrary

$$\frac{\partial}{\partial \tau} J\left(\frac{M,N}{a,b}\right) = 0$$

but $\frac{\partial S}{\partial \tau} = \frac{\partial}{\partial \tau}\left(\frac{\partial S}{\partial c}\right) = 0$ and so

$$\frac{\partial}{\partial \tau}\left\{J\left(\frac{M,N}{a,b}\right)\frac{\partial S}{\partial c}\right\} = \frac{\partial}{\partial \tau} J\left(\frac{M,N,S}{a,b,c}\right) = \frac{\partial}{\partial \tau}\left\{\alpha J\left(\frac{M,N,S}{x,y,z}\right)\right\} = 0 \qquad - (89)$$

using the fact that S is a function of c alone. Equation (89) expresses the conservation of potential vorticity (q) following fluid parcels. q may be expressed in (X,Y,Z) coordinates by noting that:

$$q = \alpha\, J\left(\frac{M,N,S}{x,y,z}\right) = \alpha J\left(\frac{X,Y,Z}{x,y,z}\right)\cdot J\left(\frac{M,N,S}{X,Y,Z}\right) = \frac{1}{\rho'}\cdot J\left(\frac{M,N,S}{X,Y,Z}\right) \quad . \qquad - (90)$$

This may be re-expressed as the determinant of a Hessian matrix by noting that eq. (63)-(65) may be written as:

$$\left.\begin{aligned} 2\Omega M &= \frac{\partial P}{\partial X} \\ 2\Omega N &= \frac{\partial P}{\partial Y} \\ \text{and} \quad gS &= \frac{\partial P}{\partial Z} \qquad \text{(letting } S = \theta/\theta_o\text{)} \end{aligned}\right\} \qquad -(91)$$

where $P = \Phi + 2\Omega^2(X^2 + Y^2)$ so that eq. (90) becomes:

$$q = \frac{\det.\underline{\underline{Q}}}{4\Omega^2\rho' g} \qquad - (92)$$

where $$\underline{\underline{Q}} = \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix} \quad .$$

(e) Geostrophic momentum coordinates

Much of the interest in semi-geostrophic theory stems from the simplified form they take when expressed in the geostrophic momentum space (M,N,Z) and from their relation to the quasi-geostrophic equations (Hoskins, 1975). Using such a coordinate transformation, Hoskins and Bretherton (1972) demonstrated the formation of a frontal discontinuity in finite time. The ageostrophic motion crucial to rapid frontogenesis is implicit in the coordinate transformation.

Consider the modified geopotential function Φ_* given by:

$$\Phi_* = \Phi + \frac{1}{2}(M - 2\Omega X)^2 + \frac{1}{2}(N - 2\Omega Y)^2 \qquad - (93)$$

and regard it as a function of M,N,Z and T (=t) with

$$X = X(M,N,Z,T)$$

$$Y = Y(M,N,Z,T)$$

Differentiating Φ_* with respect to M gives:

$$\frac{\partial \Phi_*}{\partial M} = \frac{\partial X}{\partial M}\frac{\partial \Phi}{\partial X} + \frac{\partial Y}{\partial M}\frac{\partial \Phi}{\partial Y} + (M-2\Omega X)\left(1-2\Omega\frac{\partial X}{\partial M}\right) - 2\Omega(N-2\Omega Y)\frac{\partial Y}{\partial M}$$

$$= \frac{\partial X}{\partial M}\left\{\frac{\partial \Phi}{\partial X} - 2\Omega(M-2\Omega X)\right\} + \frac{\partial Y}{\partial M}\left\{\frac{\partial \Phi}{\partial Y} - 2\Omega(N - 2\Omega Y)\right\} + M-2\Omega X$$

but the bracketed expressions are both zero by virtue of eqs (63) and (64) and so:

$$\frac{\partial \Phi_*}{\partial M} = M - 2\Omega X \; ; \qquad - (94)$$

similarly $\frac{\partial \Phi_*}{\partial N} = N - 2\Omega Y$ and $\frac{\partial \Phi_*}{\partial Z} = gS$.

Eq. (90) may be written as:

$$\frac{1}{q} = \rho' J\left(\frac{X,Y,Z}{M,N,S}\right) = \rho' J\left(\frac{M,N,Z}{M,N,S}\right) \cdot J\left(\frac{X,Y,Z}{M,N,Z}\right)$$

$$= \frac{\rho'}{\left(\frac{\partial S}{\partial Z}\right)_{M,N}} J\left(\frac{X,Y}{M,N}\right)$$

which, using eqs (94), may be rearranged and expressed as the following partial differential equation in Φ_*

$$\Phi_{*MM} + \Phi_{*NN} - \Phi_{*MM}\Phi_{*NN} + (\Phi_{*MN})^2 + \frac{4\Omega^2}{q\rho'} \Phi_{*ZZ} = 1 \qquad - (95)$$

as found by Hoskins (1975) for the f-plane case.

Furthermore, the material derivative in the geostrophic momentum space is

$$\frac{\partial}{\partial T} + \frac{DM}{Dt}\frac{\partial}{\partial M} + \frac{DN}{Dt}\frac{\partial}{\partial N} + w\frac{\partial}{\partial Z}$$

which, in view of the relations

$$\frac{DM}{Dt} = -2\Omega \frac{\partial \Phi_*}{\partial N}$$

$$\frac{DN}{Dt} = 2\Omega \frac{\partial \Phi_*}{\partial M} \qquad - (96)$$

deduced from eqs (58), (59) and (94) gives:

$$\frac{D}{Dt} = \frac{\partial}{\partial T} + 2\Omega J\left(\frac{\Phi_*}{M,N}\right) + w\frac{\partial}{\partial Z}$$

and, provided that ρ' can be expressed as a function of M,N,Z and T, potential vorticity and entropy conservation give two equations in Φ_* and w alone.

Potential vorticity conservation can also be deduced directly from the Hamiltonian form of eqs (96). As shown in Shutts and Cullen (1987) (and see below), the vector field (DM/Dt,DN/Dt,DS/Dt) is non-divergent in (M,N,S) space so that the ratio of the volume of a certain region in this space to the mass of the corresponding region in physical space, is a Lagrangian conservation property. In the limit of infinitesimal volume, this is simply the product of the specific volume and Jacobian of transformation, i.e. the potential vorticity.

Eqs (96) express the fact that in (M,N,Z) space (with Cartesian representation) the velocity (DM/Dt,DN/Dt,0) is non-divergent in Z surfaces i.e.

$$\frac{\partial}{\partial M}\left(\frac{DM}{Dt}\right) + \frac{\partial}{\partial N}\left(\frac{DN}{Dt}\right) = 0$$

The same is true in (M,N,S) space since we require from eqs (58) and (59) that:

$$\frac{\partial y}{\partial M} - \frac{\partial x}{\partial N} = 0 \qquad \text{(differentiation holding S constant)} \qquad - (97)$$

or $$- J\left(\frac{N,Y,S}{M,N,S}\right) + J\left(\frac{X,M,S}{M,N,S}\right)$$

$$= \frac{1}{\rho' q}\left[J\left(\frac{X,M,S}{X,Y,Z}\right) - J\left(\frac{N,Y,S}{X,Y,Z}\right)\right] = 0 \qquad - (98)$$

That eq. (98) is true can easily be verified by substitution for M,N and S using eqs. (91).

Purser and Cullen (1987) call this combination of the geostrophic momentum coordinates and isentropic coordinates the 'dual space' and show how the potential vorticity equation can be written in terms of a single prognostic variable. This is obtained by defining a scalar $\hat{\chi}$ such that:

$$2\Omega x = \frac{\partial \hat{\chi}}{\partial M} \; ; \; 2\Omega y = \frac{\partial \hat{\chi}}{\partial N} \text{ and } gz = \frac{\partial \hat{\chi}}{\partial S} \qquad - (99)$$

and noting that:

$$d\hat{\chi} = 2\Omega(x\,dM + y\,dN) + gz\,dS$$

$$= d(2\Omega(Mx + Ny) + gzS) - [2\Omega(M\,dx + N\,dy) + gS\,dz]$$

or, using eqs. (91):

$$d\hat{\chi} = d(2\Omega(Mx + Ny) + gzS - P) \quad . \qquad - (100)$$

Now eqs. (58) and (59) may be written in terms of $\hat{\chi}$ using eq. (99) so that:

$$\frac{DN}{Dt} = 2\Omega \left(M - \frac{\partial \hat{\chi}}{\partial M}\right) = - 2\Omega \frac{\partial \Psi}{\partial M}$$

$$\text{and} \quad \frac{DM}{Dt} = 2\Omega \left(\frac{\partial \hat{\chi}}{\partial N} - N\right) = + 2\Omega \frac{\partial \Psi}{\partial N} \qquad - (101)$$

where $\Psi = [\hat{\chi} - {}^1/_2(M^2 + N^2)]$. Furthermore using eq. (100) and the definition of P ($= \Phi + 2\Omega^2(X^2 + Y^2)$), Ψ may be equated to an energy function thus

$$\Psi = - [\Phi + 2\Omega^2(x^2 + y^2) - 2\Omega(Mx + Ny) - gzS + {}^1/_2 (M^2 + N^2)]$$

$$= - [\Phi + {}^1/_2(M - 2\Omega x)^2 + {}^1/_2 (N - 2\Omega y^2)^2 - gzS]$$

$$= - (\Phi + E) \qquad - (102)$$

where E is the energy function of the Hoskins/Bretherton equations. The reciprocal of the potential vorticity can then be expressed in terms of Ψ since:

$$\frac{1}{q} = \rho' \cdot J \left(\frac{X,Y,Z}{M,N,S}\right) = \frac{\rho'}{4\Omega^2 g} \cdot \begin{vmatrix} 1 + \Psi_{MM} & \Psi_{MN} & \Psi_{MS} \\ \Psi_{NM} & 1 + \Psi_{NN} & \Psi_{NS} \\ \Psi_{SM} & \Psi_{SN} & \Psi_{SS} \end{vmatrix}$$

and the potential vorticity equation for adiabatic flow becomes:

$$\frac{\partial}{\partial T}\left({}^{1}/_{q}\right) - 2\Omega J\left(\frac{\Psi,\ {}^{1}/_{q}}{M,N}\right) = 0 \quad . \qquad - (103)$$

The comparitive simplicity of this equation in isentropic, geostrophic momentum coordinates has been noted by others including Hoskins and Draghici (1977), Gent and McWilliams (1983) and Purser and Cullen (1987) though strictly for the f-plane system.

3. Planetary SG eigenfunctions and their accuracy

To provide some quantitative measure of the distortion inherent in the planetary semi-geostrophic equations some standard eigenvalue problems in the spherical domain have been solved using linearized equations, and compared to eigensolutions of the corresponding primitive equation problems. Two physical problems are examined; non-divergent Rossby-Haurwitz waves with a barotropic basic state atmosphere at rest and stationary planetary Rossby waves for a uniformly-stratified atmosphere in solid rotation.

a. Rossby-Haurwitz waves

Non-divergent, barotropic flow on the sphere possesses exact travelling wave solutions whose streamfunction takes the form of a spherical harmonic. The linearized planetary semi-geostrophic equations (in spherical polar coordinates) corresponding to this problem are:

$$\mathrm{Sin}^2\phi \frac{\partial v}{\partial t}_g + 2\Omega u' \mathrm{Sin}\phi = 2\Omega u_g \mathrm{Sin}\phi \qquad - (104)$$

$$\frac{\partial u}{\partial t}_g - 2\Omega v' \mathrm{Sin}\phi = -2\Omega \mathrm{Sin}\phi v_g \qquad - (105)$$

$$\frac{\partial u'}{\partial \lambda} + \frac{\partial (v' \mathrm{Cos}\phi)}{\partial \phi} = 0 \qquad - (106)$$

where $u_g \mathrm{Sin}\phi = -\frac{1}{2\Omega a}\frac{\partial \Phi'}{\partial \phi}$ and $v_g \mathrm{Sin}\phi = \frac{1}{2\Omega a \mathrm{Cos}\phi}\frac{\partial \Phi'}{\partial \lambda}$.

Eqs (104) and (105) give expressions for u' and v' (respectively) in terms of the perturbation geopotential Φ'; substitution into eq. (106) then gives a partial differential equation in Φ'. Assume now that Φ' is a travelling wave of the form

$$\Phi' = \underline{\mathrm{Re}}\left[G_m(\mu) e^{i(m\lambda - \sigma t)}\right]$$

where $G_m(\mu)$ is the wave amplitude and $\mu = \mathrm{Sin}\phi$; then it can be shown that $G_m(\mu)$ satisfies the equation:

$$(1 - \mu^2)\frac{d^2 G_m}{d\mu^2} - \frac{2}{\mu}\frac{dG_m}{d\mu} + \left(\alpha_m - \frac{m^2}{1-\mu^2}\right)G_m = 0 \qquad - (107)$$

where $\alpha_m = m^2 - 2\Omega m/\sigma$. Since $G_m = 0$ for $\mu = \pm 1$, eq. (107) constitutes an eigenvalue problem where α_m is the eigenvalue. Eigenfunctions may be expanded in normalized associated Legendre functions $P_n^m(\mu)$ so that

$$G_m(\mu) = \sum_{n=m}^{n_*} A_n^m \; P_n^m(\mu) \qquad - (108)$$

Substitution of (108) into (107) and use of the standard formulae,

$$(1 - \mu^2)\frac{d^2P_n^m}{d\mu^2} - 2\mu\frac{dP_n^m}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right] P_n^m = 0$$

$$(1 - \mu^2)\frac{dP_n^m}{d\mu} = (n+1)\,\varepsilon_n^m\, P_{n-1}^m - n\varepsilon_{n+1}^m\, P_{n+1}^m$$

and $$\mu P_n^m = \varepsilon_{n+1}^m\, P_{n+1}^m + \varepsilon_n^m P_{n-1}^m$$

where $\varepsilon_n^m = \left(\frac{4n^2 - 1}{n^2 - m^2}\right)^{1/2}$ gives, on collecting terms:

$$\sum_{n=m}^{n_*} A_n^m \left\{\varepsilon_{n+1}^m[\alpha_m + n(1-n)]P_{n+1}^m + \varepsilon_n^m[\alpha_m - (n+1)(n+2)]P_{n-1}^m\right\} = 0 \quad .$$

The orthogonality of P_n^m then implies a two-term recurrence relation

$$A_{s-1}^m\, \varepsilon_s^m\, \Gamma_{s-2}^m + A_{s+1}^m\, \varepsilon_{s+1}^m\, \Gamma_{s+2}^m = 0 \qquad s = m, m+1.... \qquad - (109)$$

where $\Gamma_s^m = m^2 - 2\Omega m/\sigma - s(s+1)$. If the function $\Gamma_r^m = 0$ for some r

then it can be shown that all of the coefficients vanish except

A_{r-1}^m and A_{r+1}^m giving a set of eigenfunctions $G_m^{(r)}(\mu)$ with

$$G_m^{(r)}(\mu) = A_{r-1}^m\, P_{r-1}^m + A_{r+1}^m\, P_{r+1}^m \qquad - (110)$$

and $$\sigma = \frac{-2\Omega m}{r(r+1) - m^2} \quad . \qquad - (111)$$

The dispersion relation (111) should be contrasted with the corresponding equation for the Rossby-Haurwitz wave ie

$$\sigma = \frac{-2\Omega m}{r(r+1)} \quad .$$

Let $v'\cos\phi = imV'(\mu)\exp i(m\lambda - \sigma t)$ - (112)

and use eq. (105) to obtain an expression for V' in terms of $G(\mu)$, then it may be shown that:

$$V'(\mu) = \frac{1}{2\Omega a}\left[\frac{G}{\mu} + \frac{(1-\mu^2)}{\mu^2}\left(\frac{\sigma}{2\Omega m}\right)\frac{dG}{d\mu}\right] \quad \text{- (113)}$$

Using eqs.(109), (110), the condition $\Gamma_r^m = 0$ and the recurrence formulae for the associated Legendre function, it can be shown that:

$$V'(\mu) = \frac{A_{r+1}^m \; P_r^m(\mu)}{2\Omega a \varepsilon_{r+1}^m} \quad \text{- (113)}$$

The continuity eq. (106) may be written as:

$$\frac{\partial(u'\cos\phi)}{\partial\lambda} + (1 - \mu^2)\frac{\partial(v'\cos\phi)}{\partial\mu} = 0$$

implying a streamfunction ζ satisfying the relations

$$u'\cos\phi = -(1-\mu^2)\frac{\partial\zeta}{\partial\mu} \quad \text{and} \quad v'\cos\phi = \frac{\partial\zeta}{\partial\lambda} \; .$$

Using eqs.(112) and (114) it follows that

$$\zeta = \frac{A_{r+1}^m}{2\Omega a \varepsilon_{r+1}^m} P_r^m(\mu)\exp\left[i(m\lambda - \sigma t)\right]$$

- exactly the same spherical harmonic form as the Rossby-Haurwitz wave though with the distorted phase speed formula eq. (111). Considering the case m=1, the angular phase speed (σ/m) is grossly in error only for the gravest planetary mode r=1 for which the phase speed is -2Ω rather than the true value of $-\Omega$. For r=2 σ/m is $-2\Omega/5$ rather than the true value of $-\Omega/3$, and r=3 gives $-2\Omega/11$ rather than $-\Omega/6$: convergence to the true phase speed is rapid. The accuracy of the modes with $r >> m$ is clearly related to the elongation of the implied eddy circulations in the zonal direction so that the contribution of the zonal flow components to the total kinetic energy outweighs that of the meridional components (wherein lies the main approximation in the derivation of the planetary semi-geostrophic equations). For higher zonal wavenumbers the error in the gravest meridional mode (m=r) becomes relatively worse since the eddy circulations are then elongated in the meridional direction. In fact, for large r, the angular phase speed of these modes tends to $-2\Omega/r$ rather than the correct value of $-2\Omega/r^2$.

For at least two reasons this Rossby-Haurwitz wave test of the accuracy of the PSG equations is unduly severe and should not be taken on its own as a measure of the practical value of the equation set. Firstly, the Rossby-Haurwitz wave is not an accurate model of large-scale travelling wave motion. The rate of retrogression (westward propagation relative to the airflow) is far greater than is observed due primarily to the neglect of divergence and the associated vertical structure of real planetary Rossby waves. The total energy

associated with these waves involves a potential energy component as well as the kinetic energy; their ratio is given by the Burger number defined in Section 2(b). Therefore, the error introduced through the neglect of $^1/_2\ (v\cos\phi)^2$ from the Hamiltonian should be measured against the sum of the remaining kinetic energy ($^1/_2(v\sin\phi)^2 + {}^1/_2u^2$) and the potential energy of the wave. As shown earlier, for motions with small Burger number the entire kinetic energy may be neglected and the Phillips Type II equations obtained.

To assess the accuracy of the planetary semi-geostrophic equations in a more realistic context we look at the properties of stationary, three-dimensional planetary Rossby waves and compare PSG solutions with the corresponding primitive equation solutions.

b. Stationary, baroclinic planetary waves

Assume, for simplicity, small amplitude sinusoidal perturbations of a basic state atmosphere in solid rotation with zonal velocity $\bar{U}\cos\phi$ (where $\bar{U}$ is constant) and with potential temperature increasing linearly with Z so that:

$\theta = \theta_o\ (1 + \bar{B}Z)$ where $\bar{B}$ is the static stability;

the zonal wind is supported by a geopotential $\bar{\Phi}$ given by:

$$\bar{U}\ \mathrm{Cos}\phi = -\frac{1}{2\Omega a\ \sin\phi}\frac{d\bar{\Phi}}{d\phi} \qquad \text{- (115)}$$

Denoting the wave components by primes we have

$$u = \bar{U}\cos\phi + u'$$

$$v = v'$$

$$w = w'$$

$$\Phi = \bar{\Phi} + \Phi'$$

$$\left.\begin{aligned} u_g &= \bar{U}\,\mathrm{Cos}\phi - \frac{1}{2\Omega a \sin\phi}\frac{\partial\Phi'}{\partial\phi} \\ v_g &= \frac{1}{2\Omega a \sin\phi \mathrm{Cos}\phi}\frac{\partial\Phi'}{\partial\lambda} \\ \theta &= \theta_o(1 + \bar{B}Z) + \frac{\theta_o}{g}\frac{\partial\Phi'}{\partial Z} \end{aligned}\right\} \quad - (116)$$

(consistent with eq. 82, and dropping the LV subscript from now on so that $\partial/\partial Z$ represents $(\partial/\partial Z)_{LV}$)

where the geostrophic wind and hydrostatic relations have been used. Substituting the required eqs. from (116) into the PSG momentum equations (80) and (81) with $\partial/\partial t \equiv 0$ and neglecting products of primed variables gives:

$$2\Omega v' \mathrm{Sin}\phi = \frac{1}{a\mathrm{Cos}\phi}\frac{\partial\Phi'}{\partial\lambda} - \frac{\bar{U}}{2\Omega a^2 \mathrm{Sin}\phi}\frac{\partial^2\Phi'}{\partial\lambda\partial\phi} \quad - (117)$$

$$2\Omega u' \mathrm{Sin}\phi = -\frac{1}{a}\frac{\partial\Phi'}{\partial\phi} - \frac{\bar{U}\,\mathrm{Tan}\phi}{2\Omega a^2}\frac{\partial^2\Phi'}{\partial\lambda^2} \quad - (118)$$

where some factors of $(1 \pm \bar{U}/2\Omega a)$ have been replaced with unity for convenience in view of the fact that $|\bar{U}|/2\Omega a \ll 1$ for typical atmospheric values of $\bar{U}$. The conservation of potential temperature can similarly be shown to reduce to:

$$w' = -\frac{\bar{U}}{ag\bar{B}} \frac{\partial^2 \Phi'}{\partial Z \partial \lambda} \quad . \qquad - (119)$$

Eqs. (117) - (119) give expressions for u', v' and w' in terms of the perturbation geopotential Φ': these may be linked through the continuity equation for quasi-incompressible flow

$$\frac{1}{a\cos\phi} \left(\frac{\partial u'}{\partial \lambda} + \frac{\partial}{\partial \phi}(v'\cos\phi) \right) + \frac{\partial w'}{\partial Z} = 0 \qquad - (120)$$

(Note that Z is essentially a pressure coordinate and so air parcels may expand/contract slightly following the motion.)

We seek stationary planetary Rossby wave solutions with sinusoidal variation in λ and Z which tilt westwards with height. These correspond to modes which transmit wave energy upwards and would arise in problems with lower tropospheric forcing such as sinusoidal orography.

Let $\Phi' = G_m(\phi')\exp[i(m\lambda+\nu z)]$ where ϕ' is the co-latitude; substitute eqs. (117) - (119) into (120) and rearrange into an ordinary differential equation for $G_m(\phi')$ so that:

$$\frac{d^2G_m}{d\phi'^2} + \frac{2 - \cos^2\phi'}{\cos\phi'\sin\phi'}\frac{dG_m}{d\phi'} - \left(\frac{m^2}{\sin^2\phi'} - \alpha_m\right)G_m = \hat{\lambda}\cos^2\phi G_m \qquad - (121)$$

where $\alpha_m = m^2 + 2\Omega a/\bar{U}$ and the eigenvalue $\hat{\lambda}$ corresponds to $(2\Omega a\nu)^2/g\bar{B}$. Upward energy propagation corresponds to $\nu^2>0$ - otherwise the modes are evanescent. The eigenvalue problem for $G_m(\phi')$ and $\hat{\lambda}$ requires the specification of polar boundary conditions. Orszag (1974) showed that the following natural boundary condition is appropriate

$$\frac{d^kG_m(\phi')}{d\phi'^k} = 0 \qquad \text{at } \phi' = 0, \pi \quad \text{for } k = 0, 1, \ldots |m| - 1.$$

It is not necessary to satisfy these constraints precisely given a truncated spectral or pseudo-spectral expansion of the function G_m (Boyd, 1978). All that is required is that the error in satisfying these conditions can be reduced indefinitely by taking enough terms in the expansion. The pseudo-spectral approach (using modified Fourier basis functions) recommended by Boyd (1978) was used to solve the eigenvalue problem. The modified Fourier basis functions $\theta_j(\phi')$ are given by:

$$\theta_j(\phi') = \begin{cases} \sin\phi'\cos(j\phi') & m \text{ odd} \\ \cos(j\phi') & m \text{ even} \end{cases}$$

and are associated with a weighting function $w_i = \delta(\phi'_j)$ (δ is the Dirac delta function) with

$$\phi'_j = \frac{2\Pi j}{n_*} \quad \text{and with } G_m(\phi') = \sum_{n=0}^{n_*-1} a_n \Theta_n(\phi') \qquad - (122)$$

where n_* is the number of terms in the expansion. If eq. (121) is written as

$$L(G_m) = 0$$

where L is the implied differential operator, then the pseudo-spectral method gives n_* equations

$$\sum_{j=0}^{n_*-1} \left\{ \int_o^{\pi} w_i L(\theta_j) \mathrm{Sin}\phi' d\phi' \right\} a_j = 0 \qquad i = 0, n_* - 1 \qquad - (123)$$

which may be written as a matrix equation of the form:

$$\underline{\underline{D}}\ \underline{A} = \hat{\lambda}\ \underline{\underline{E}}\ \underline{A} \qquad - (124)$$

where $\underline{A}$ is the coefficient vector whose components are a_o, a_1 ... a_{n*-1} and $\underline{\underline{D}}$ and $\underline{\underline{E}}$ are two matrices. The matrix eigenvalue problem (124) was solved using a standard routine in the NAG mathematical subroutine library. $G_m(\phi')$ can be obtained by substituting the a_n values into eq. (122) and evaluating the sum. The vertical wavelength l_z of the stationary Rossby wave can be obtained from $\hat{\lambda}$ using

$$l_z = \frac{2\pi}{\nu} = \frac{4\pi\Omega a}{gB^{1/2}\hat{\lambda}^{1/2}} \qquad - (125)$$

Primitive equation solutions corresponding to the above PSG eigenvalue problem were obtained by the pseudo-spectral technique as used by Ahlquist (1982) for travelling planetary scale waves. Letting

$$u' = (\Omega a)\,\hat{U}(\phi')\exp[i(m\lambda + \nu z)]$$

$$v' = -im(\Omega a)\,\hat{V}(\phi')\exp[i(m\lambda + \nu z)]$$

and $$\Phi' = (\Omega^2 a^2)\,\hat{\Phi}(\phi')\exp[i(m\lambda + \nu z)]$$

the momentum and continuity equations may be written as:

$$\varepsilon\hat{U} + 2\,\mathrm{Cos}\phi'\hat{V} = -\frac{\hat{\Phi}}{\mathrm{Sin}\phi'}, \qquad - (126)$$

$$m^2\varepsilon\hat{V} + 2\mathrm{Cos}\phi'\hat{U} = \frac{d\hat{\Phi}}{d\phi'}, \qquad - (127)$$

and $$\hat{U} + \frac{d}{d\phi}[\hat{V}\mathrm{Sin}\phi'] + \kappa\hat{\Phi}\,\mathrm{Sin}\phi' = 0 \qquad - (128)$$

where the vertical velocity w in the continuity equation has been eliminated using the thermodynamic equation (119) and κ is given by:

$$\kappa = \varepsilon(\Omega a\nu)^2/g\bar{B} \text{ and } \varepsilon = U/2\Omega a \quad . \qquad - (129)$$

$\hat{U}$, $\hat{V}$ and $\hat{\Phi}$ are expanded so that:

$$(\hat{U}, \hat{V}) = \sum_{n=0}^{n_*-1} (U_n, V_n)\,\mathrm{Sin}^{|k-1|}\phi'.\mathrm{Cos}n\phi' \qquad - (130)$$

$$\hat{\Phi} = \sum_{n=0}^{n_*-1} \Phi_n\,\mathrm{Sin}^k\phi'.\mathrm{Cos}n\phi'$$

where $k = \begin{cases} 0 & \text{for even } m \\ 1 & \text{for odd } m \end{cases}$

and $\int_0^{\pi} \delta(\phi'_i)$[eqs (126)-(128)]$d\phi'$, i = 0, n_*-1 provides $3n_*$ equations for the $3n_*$ unknown amplitude coefficients in eqs (130). These may be written in the same matrix form as eq. (124) and solved for the eigenfunctions and eigenvalues. For both PSG and primitive equations the following parameter values were chosen:

$$\Omega = 7.292 \times 10^{-5} \ s^{-1}$$

$$a = 6.371 \times 10^{6} \ m$$

$$\bar{U} = 14.14 \ ms^{-1}$$

$$g\bar{B} = 1 \times 10^{-4} \ s^{-1}$$

and eigenvalues were re-expressed in terms of the corresponding vertical wavelengths. The resulting eigenfunctions were normalised so that:

$$\int_0^{\Pi} G^2_m \ \mathrm{Sin}\phi' d\phi' = 1$$

Figs. 2 (a)-(d) show the gravest antisymmetric geopotential perturbations from the PSG equations (solid line) and primitive equations (dashed line) for wavenumbers 1, 3, 5 and 7 respectively.

Also indicated are the corresponding vertical wavelengths with imaginary values representing decay lengths. The overall comparison is excellent: only wavenumber 7 shows signs of substantial discrepancy. An equatorially trapped, though vertically propagating (ν real), m=7 mode was found in the primitive equation calculation which had no counterpart in the PSG eigenvalue problem. Further linear analysis of the planetary semi-geostrophic equations is beyond the scope of this paper though highly desirable. In particular it would be interesting to study baroclinic instability on the sphere to see if the level of distortion in the PSG equations is acceptable. Observed baroclinic weather systems tend to be elongated in the meridional direction (Hoskins et al, 1983) and are characterized by a Burger number of order unity. The error in approximating the Hamiltonian is therefore unlikely to be negligible in baroclinic instability studies.

4. Discussion

In Section 2 it was shown that by approximating Hamilton's principle for a perfect fluid in a way which results in new canonical coordinates, a simple set of filtered equations of motion- consistent with the principle - can be derived. These equations automatically have analogues of the conservation properties of the unapproximated equations such as global energy conservation and Lagrangian conservation of potential vorticity. It was also shown that the Burger equations (Phillips quasi-geostrophic type II) are obtained if the kinetic energy is omitted from the Hamiltonian. X and Y are then the canonical coordinates and the equations of motion are (rewriting eqs (33) and 34) simply:

$$\frac{\partial Y}{\partial \tau} = \frac{1}{2\Omega}\frac{\partial \Phi}{\partial X}$$

and $$\frac{\partial X}{\partial \tau} = -\frac{1}{2\Omega}\frac{\partial \Phi}{\partial Y}$$

- ie the geostrophic wind relation. By choosing canonical coordinates $M(=2\Omega x+v)$ and $N(=2\Omega y-u)$, and neglecting both a term representing the ratio of the Lagrangian rate of turning of the wind direction to 2Ω and the contribution of the axial motion (parallel to $\underline{\Omega}$) to the kinetic energy, the semi-geostrophic evolution equations

$$\frac{\partial N}{\partial \tau} = 2\Omega \frac{\partial \Phi_*}{\partial M}$$

$$\frac{\partial M}{\partial \tau} = -2\Omega \frac{\partial \Phi_*}{\partial N}$$

are obtained (eqs.96). In the f-plane theory, the latter assumption is extremely accurate: for planetary flow it entails the neglect of a significant amount of kinetic energy depending on the degree of zonality of the motion . Large-scale planetary motion in the terrestrial atmosphere is highly anisotropic with zonal velocity perturbations dominating meridional flow perturbations (Charney, 1971). Atmospheric flow on Jupiter and Saturn is even more highly biassed towards zonality (Ingersoll et al, 1979). Rhines (1975) and Williams (1978) have demonstrated that zonality is a natural state towards which 'turbulent' two-dimensional flows should migrate in the presence of a latitudinal gradient in the Coriolis parameter. It seems fitting, therefore, that the planetary semi-geostrophic set proposed here becomes highly accurate in this limit.

Salmon (1985) has derived a different set of semi-geostrophic equations valid for variable Coriolis parameter (f) by considering a shallow rotating layer of inviscid, homogeneous fluid in a Cartesian system with f(x,y). Although convenient, the device of variable f in a Cartesian system is rather unnatural since it ignores important metric factors which arise, for instance, if x and y are Mercator coordinates. The derivation of the PSG equations in Section 2(c) does not assume any particular geometry of the underlying solid planet as is implicit in the choice of f(x,y) in Salmon's approach. On the other hand, Salmon's system of equations is more accurate to the extent that the kinetic energy in the Hamiltonian is approximated by the total geostrophic kinetic energy whereas only part of this is retained in the PSG formulation of Hamilton's principle.

It should be noted that the f-plane limit of the PSG equations can only be considered to arise from making the tangent plane approximation at the (rotational) pole of the spherical system. This limit is not the same for tangent plane approximations at other points on the sphere due to the omission of the axial velocity contribution to the kinetic energy. The usual procedure is to set up the cartesian system with z axis oriented in the local vertical direction and to ignore the horizontal component of $\underline{\Omega}$. A more accurate semi-geostrophic set for the sphere could be derived using a spherical polar coordinate adaption of Salmon's method in which the Coriolis parameter is evaluated by replacing the physical coordinates with the geostrophic momentum coordinates in its functional dependence on position.

Since the PSG equations take exactly the same form as the f-plane semi-geostrophic equations, they may be solved using exactly the same techniques (Schubert, 1985). It would be interesting to extend the baroclinic instability studies of Hoskins and West (1979) and Hoskins and Heckley (1981) to flows with a background potential vorticity gradient using the PSG equations in geostrophic momentum coordinate form. The equations could also be used in the study of large amplitude planetary Rossby waves in the middle atmosphere for which the Burger number would be small. The time-averaged state of the terrestrial atmosphere is dominated by zonal flow and low zonal wavenumbers (m=1,2 and 3). The PSG equations could therefore form the basis of a low-order climate model of the type used by Shutts (1983) and White and Green (1982) where the time-averaged motion is represented explicitly and the dynamical effects of transient baroclinic instabilities are parametrized in terms of the transfer of potential vorticity. The existence of a diagnostic relation between the potential vorticity and Φ_* (e.g. eq. 95) is crucial to this type of parametrized climate model.

Further work is required to establish what classes of equatorially trapped motion are supported by the PSG equations. Initial investigations have shown that a form of Kelvin wave is permitted which is accurate for long zonal wavelengths.

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Legends

Fig.1 (a) Schematic diagram showing the difference between differentiation with respect to x and X.

(b) As above except for z and Z differentiation.

Fig.2 Comparison between PSG (dashed line) and primitive equation (solid line) geopotential eigenfunctions for the stationary baroclinic Rossby wave problem for zonal wavenumbers 1 (a), 3 (b), 5 (c) and 7 (d). The corresponding dimensional vertical wavelengths are also indicated. The factor of i appearing in the m=7 l_z formulae indicates that the length concerned is an exponential decay scale.

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$$\delta\Phi_{AC} = \Phi_A - \Phi_C$$

$$\frac{\partial\Phi}{\partial x} = \lim_{\delta X \to 0} \left\{ \frac{\delta\Phi_{AC}}{\delta X} \right\}$$

$$\frac{\partial\Phi}{\partial X} = \lim_{\delta X \to 0} \left\{ \frac{\delta\Phi_{AB}}{\delta X} \right\}$$

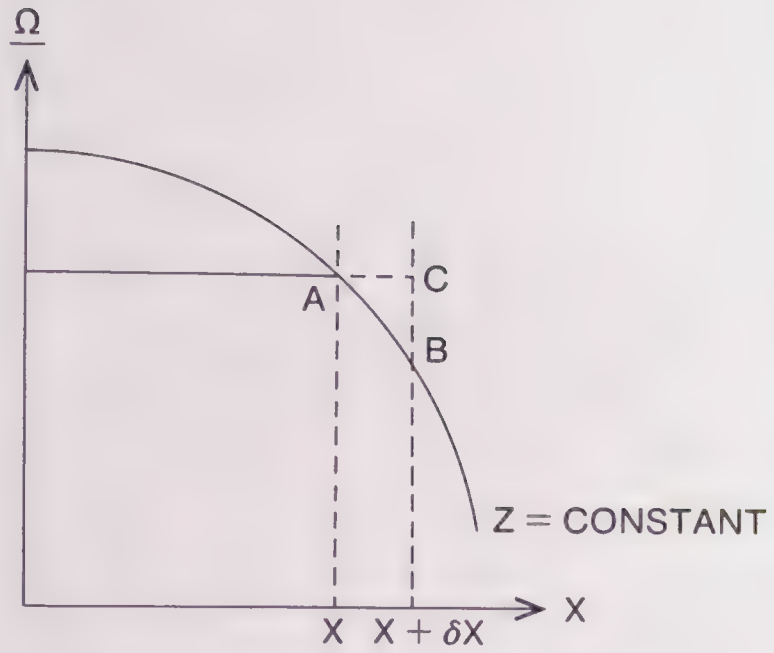


Fig. 1(a)

$$\frac{\delta\Phi_{AC}}{\delta Z} = \frac{\delta\Phi_{AB}}{\delta Z} + \frac{r\delta\phi}{\delta Z} \cdot \frac{\delta\Phi_{BC}}{r\delta\phi}$$

$$\therefore \quad \delta Z \to 0$$

$$\frac{\partial\Phi}{\partial Z} \approx \left(\frac{\partial\Phi}{\partial Z}\right)_{LV} + \frac{\cot\phi}{r} \quad \frac{\partial\Phi}{\partial\phi}$$

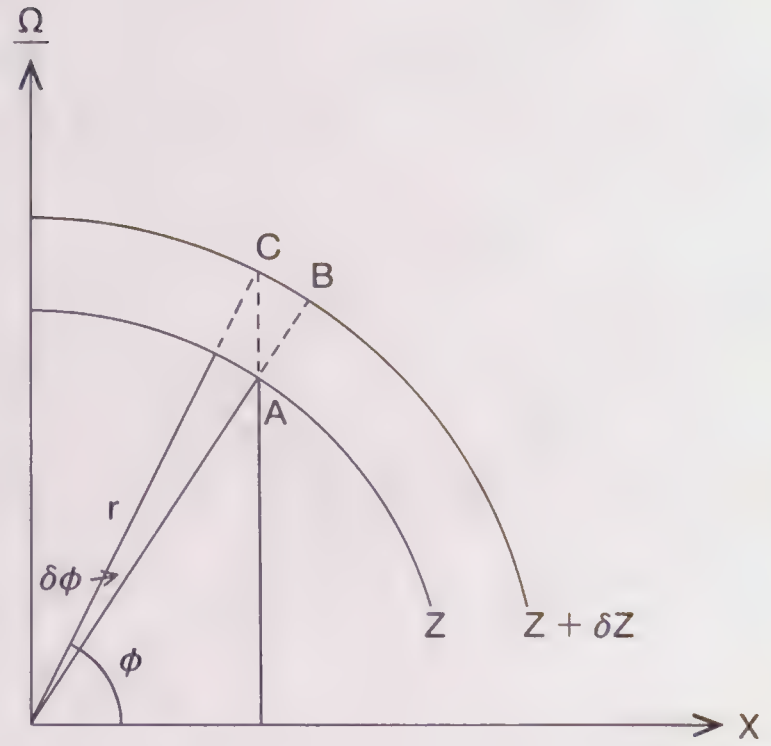


Fig. 1(b)

Fig 2 (a)

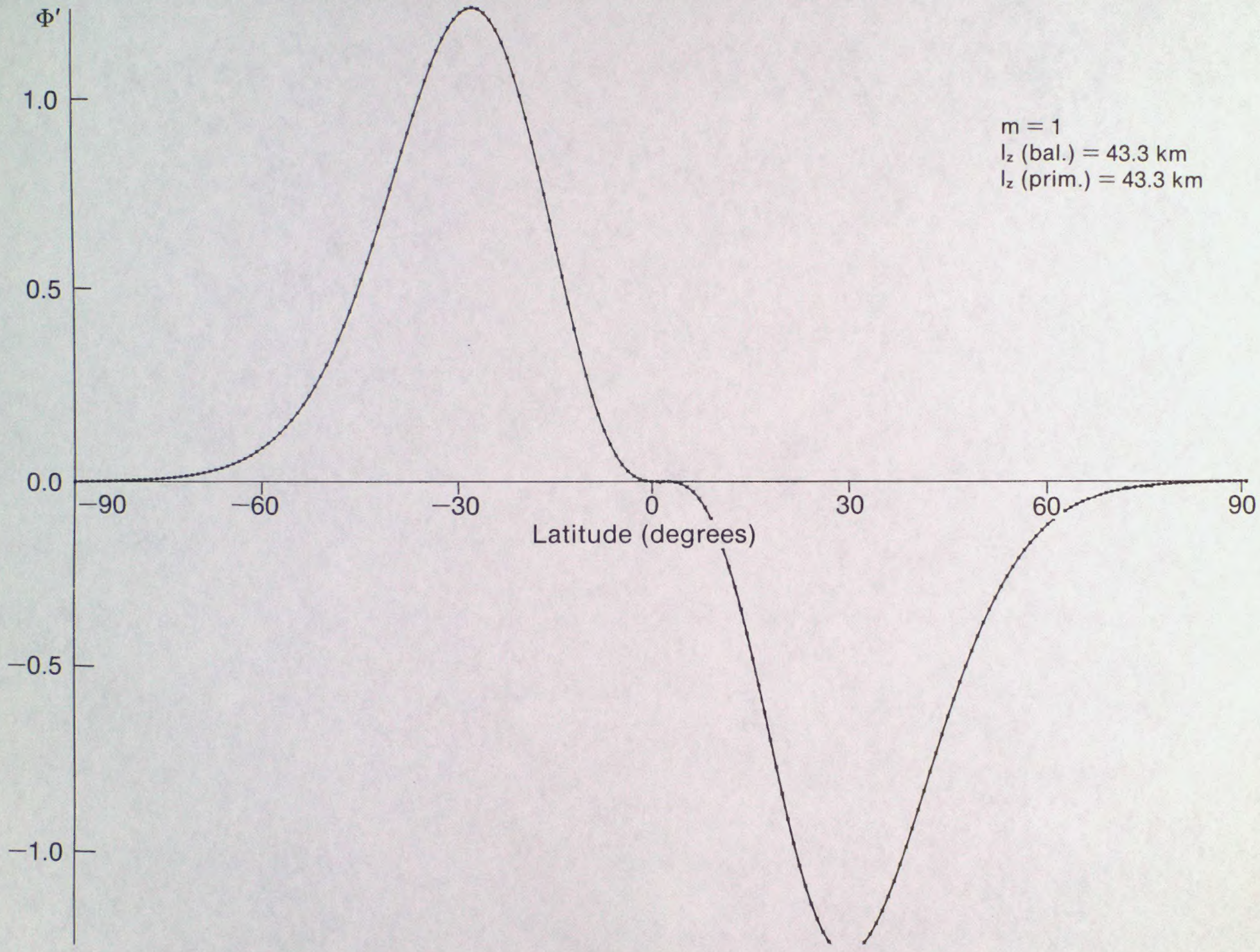
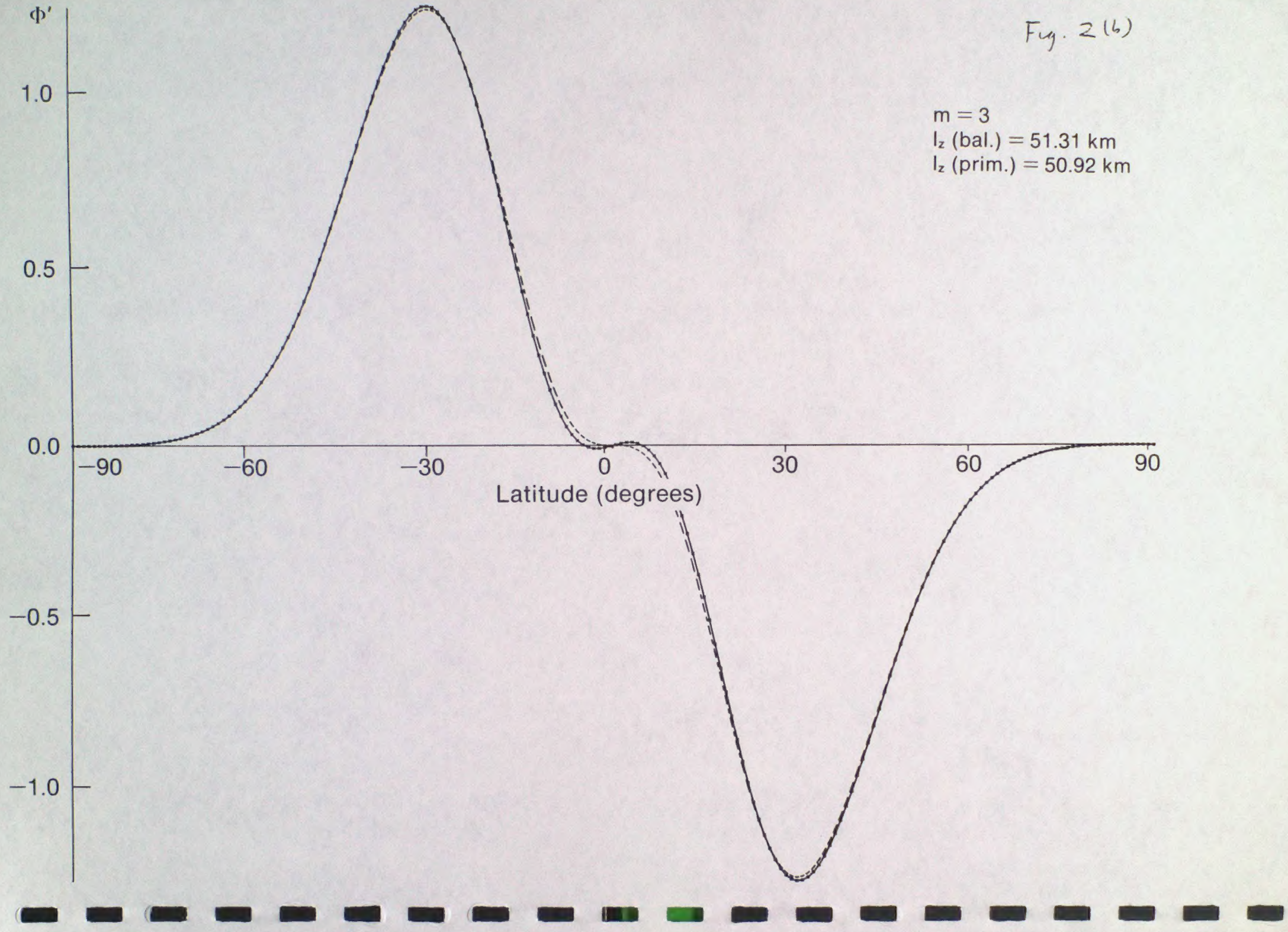

Φ′
1.0
0.5
0.0
−0.5
−1.0
−90
−60
−30
0
30
60
90
Latitude (degrees)
m = 1
l_z (bal.) = 43.3 km
l_z (prim.) = 43.3 km

Fig. 2(b)


Φ′
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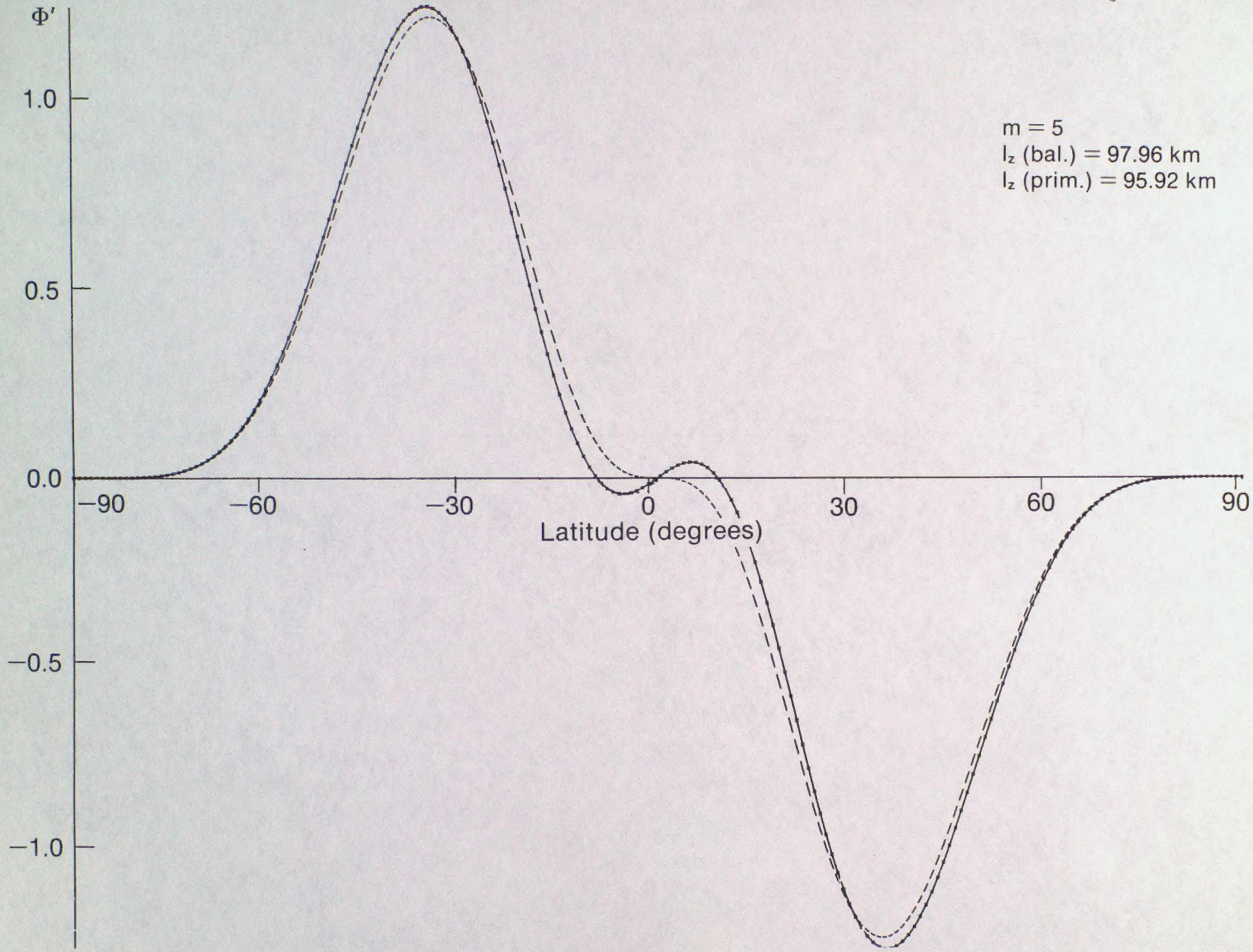
0.5
0.0
−0.5
−1.0
−90
−60
−30
0
30
60
90
Latitude (degrees)
m = 3
l_z (bal.) = 51.31 km
l_z (prim.) = 50.92 km

Fig. 2 (c)


Φ′
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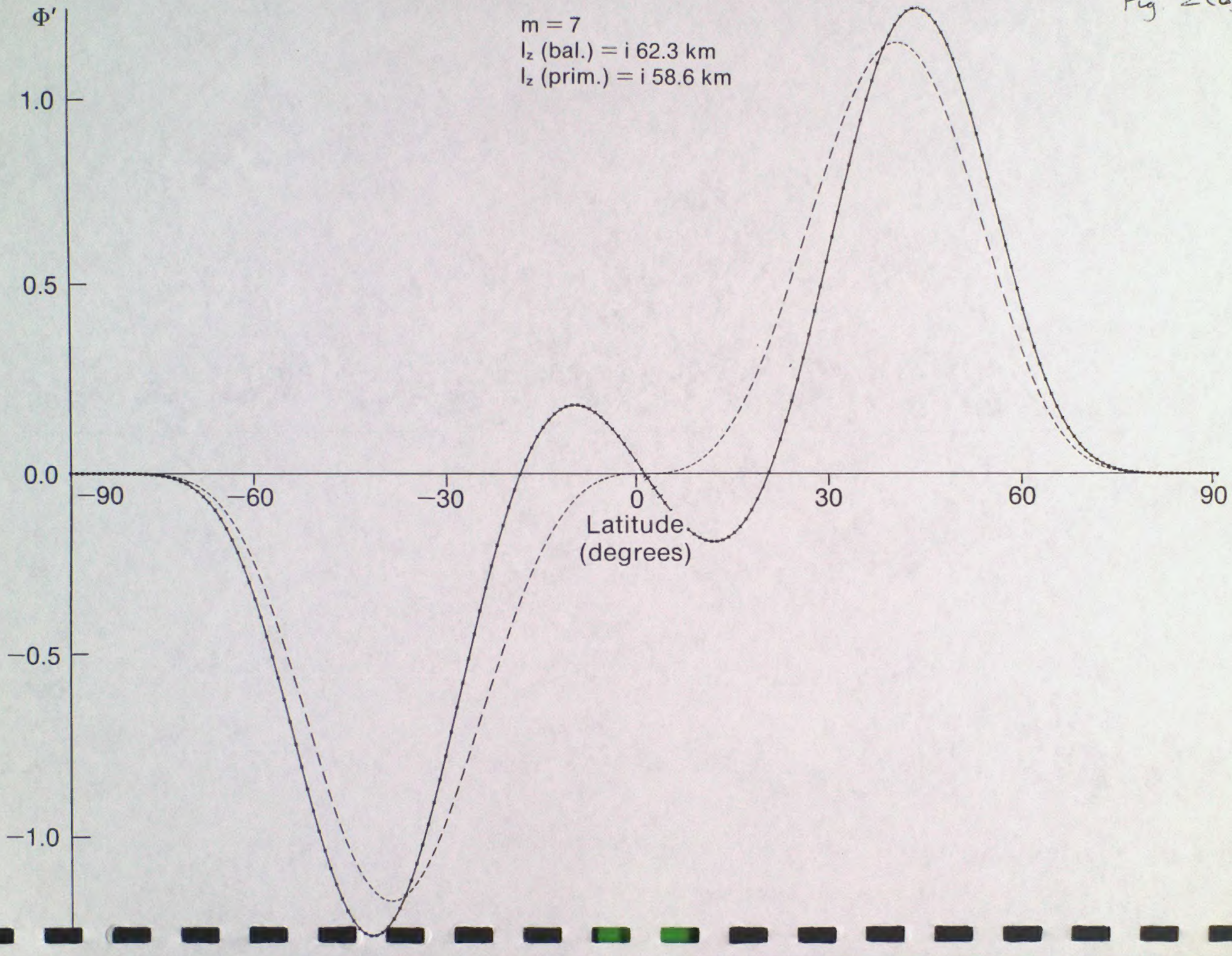
0.5
0.0
−0.5
−1.0
−90
−60
−30
0
30
60
90
Latitude (degrees)
m = 5
l_z (bal.) = 97.96 km
l_z (prim.) = 95.92 km

Fig 2(d)


Φ′
m = 7
l_z (bal.) = i 62.3 km
l_z (prim.) = i 58.6 km
1.0
0.5
0.0
−0.5
−1.0
−90
−60
−30
0
30
60
90
Latitude
(degrees)